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Singularities in solutions of the three-dimensional laminar-boundary-layer equations

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The three-dimensional steady laminar-boundary-layer equations have been cast in the appropriate form for semisimilar solutions, and it is shown that in this form they have the same structure as the semisimilar form of the two-dimensional unsteady laminar-boundary-layer equations. This similarity suggests that there may be a new type of singularity in solutions to the three-dimensional equations: a singularity that is the counterpart of the Stewartson singularity in certain solutions to the unsteady boundary-layer equations.

A family of simple three-dimensional laminar boundary-layer flows has been devised and numerical solutions for the development of these flows have been obtained in an effort to discover and investigate the new singularity. The numerical results do indeed indicate the existence of such a singularity. A study of the flow approaching the singularity indicates that the singularity is associated with the domain of influence of the flow for given initial (upstream) conditions as is prescribed by the Raetz influence principle.

1. Introduction

The variety and significance of singularities that occur in the solutions to the laminar-boundary-layer equations have only recently really begun to be recognized. The classical Goldstein singularity (Goldstein 1948) that occurs in solutions to the steady two-dimensional boundary-layer equations, subjected to an adverse pressure gradient, has, of course, been known for over thirty years. This singularity is associated with the vanishing of the wall shear in the solution, and hence with steady boundary-layer separation.

In 1951 Stewartson noted a new singularity in solutions to the unsteady boundary-layer equations. It occurs at the outer edge of the boundary layer, in contrast with the Goldstein singularity, which occurs at the wall. This singularity is found in the solution to the boundary-layer equations for a semi-infinite flat plate that is impulsively set into motion at time $t = 0$ with a uniform velocity U. Stewartson pointed out that the singularity occurs at the dimensionless time Ut/x of unity and physically represents a division between two modes of development of the boundary layer at a given point x on the plate. For $Ut/x < 1$ the flow develops locally as if it were unaware of the leading edge of the plate; for $Ut/x > 1$ the flow develops under the influence of the leading edge and ultimately, at large time, is represented by the Blasius solution. This type of singularity also appears in solutions for the flow past wedges that are impulsively set into motion, where it has the same physical significance (Nanbu 1977; Smith 1967; Williams & Rhyme 1980).

In recent years numerical investigations of the unsteady laminar-boundary-layer equations have indicated that there is another singularity in the solutions to the equations when the flow develops under the influence of an adverse pressure gradient (Telionis, Tsahalis & Werle 1973, Williams & Johnson 1974, 1975; Williams 1982). These results further indicate that as the singularity is approached the calculated characteristics of the boundary layer approach those associated with unsteady separation as postulated in the Moore–Rott–Sears model. In the Moore–Rott–Sears model for unsteady separation, the velocity and shear vanish simultaneously at an interior point of the boundary layer, as seen in a coordinate system that is moving with separation. Quite recently, Williams & Stewartson (1982) have established the structure of this singularity, at least the case where separation is associated with the impulsive motion of a body with a sharp trailing edge.

A singularity is also found in numerical solutions to the steady three-dimensional laminar-boundary-layer equations (Cebeci, Khatteb & Stewartson 1982; Williams 1975). This singularity appears to be associated with three-dimensional separation, and the existing numerical solutions indicate that the line of separation is an envelope of limiting streamlines.

At this juncture, the author must point out, lest the reader misunderstand, that the singularities discussed here are not to be expected in any real physical flow. Nature simply does not allow such singularities. These singularities arise only as a result of using an incomplete set of equations, the boundary-layer equations, in an attempt to describe the physics of the flow. It is certainly expected that, if we could solve the full equations describing the flow field, the Navier–Stokes equations, the solutions would not contain these singularities. On the other hand, it is important to note that all of the singularities found in solutions to the boundary-layer equations appear to be associated with a real physical or mathematical phenomenon. In each case, the calculated flow approaching the singularity exhibits physical characteristics that are associated with a real physical phenomenon. Thus, while we do not expect to observe singularities in real physical flows, the study of flows that contain singularities (in their solutions) yield valuable insight into the physics of real flows.

In the present work we undertake a brief review of the semisimilar formulation of the two-dimensional unsteady and three-dimensional steady laminar-boundary-layer equations. In this review it is possible to identify all of the known singularities in the solutions to these equations and to identify the physical phenomenon that is associated with each singularity. We are also led to the conclusion that there is another, previously unidentified, singularity in the solutions of the three-dimensional study of laminar-boundary-layer equations. Solutions are obtained for this new case and compared with solutions for three-dimensional steady separation. The importance and meaning of the singularity are identified.

2. Semisimilar formulation

The singularities associated with a change in character of a flow impulsively set into motion (Stewartson's singularity) and the singularity associated with unsteady separation become readily apparent when the unsteady problem is formulated in a pair of scaled coordinates (a semisimilar scaling). These singularities occur when a certain coefficient in this formulation vanishes (Wang & Shen 1978; Williams 1981). We further note that a semisimilar transformation may be applied to the three-dimensional steady laminar-boundary-layer equations (Williams 1975), and, when formulated in this fashion, the separation singularity is apparent and again occurs when a certain coefficient vanishes. We therefore formulate both the two-dimensional unsteady and the three-dimensional steady problem in the semisimilar form with a view to identifying existing and new singularities.

2.1. *Two-dimensional unsteady laminar boundary layers*

We consider first the case of two-dimensional incompressible unsteady laminar boundary layers. Let x and y be the coordinates along and normal to the surface respectively, and u and v be the corresponding velocity components. Further, let t be time, ν be the kinematic velocity and u_δ be the velocity at the edge of the boundary layer. The laminar-boundary-layer equations describing the boundary-layer motion of interest are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial u_\delta}{\partial t}+u_\delta\frac{\partial u_\delta}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}. \tag{2}$$

The boundary conditions applicable to the solutions of these equations are the usual no-slip conditions at the wall and the condition that the x-component of velocity, $u(x,y,t)$ match the known inviscid-flow solution $u_\delta(x,t)$ as the distance from the wall becomes large.

A major difficulty encountered in the solution of (1) and (2) is the existence of three independent variables (x,y,t) in the problem. The technique of semisimilar solutions seeks to eliminate this difficulty by reducing the number of independent variables from three $(x,\ y,\ t)$ to two $(\eta,\ \xi)$ by appropriate scalings. In the spirit of this transformation, we introduce two new scaled coordinates η and ξ defined by

$$\eta=\frac{y(U/\nu l)^{\frac{1}{2}}}{g^*(x^*,t^*)},\quad \xi=\xi(x^*,t^*).$$

Here $x^*=x/l$ and $t^*=tU/l$ are normalized x- and time coordinates, U is a characteristic velocity and l is a characteristic length for the problem. The function $g^*(x^*,t^*)$ may be thought of as a scaling function for the normal, or y-, coordinate and $\xi(x,t)$ may be thought of as a new x-coordinate which has been scaled with time. In addition, a new non-dimensional stream function $f(\xi,\eta)$ is introduced. This non-dimensional stream function is related to the usual stream function $\Psi(x,y,t)$ by

$$\Psi(x,y,t)=(\nu Ul)^{\frac{1}{2}}g^*(x^*,t^*)\,u_\delta^*(x^*,t^*)f(\xi,\eta),$$

in which $u_\delta^*=u_\delta/U$. The continuity equation is satisfied identically by the introduction of the stream function, and the momentum equation becomes, in terms of the dimensionless stream function and new coordinates,

$$\frac{\partial^3 f}{\partial\eta^3}+(d+e)f\frac{\partial^2 f}{\partial\eta^2}+d\left(1-\left(\frac{\partial f}{\partial\eta}\right)^2\right)+a\left(1-\frac{\partial f}{\partial\eta}\right)$$
$$+\frac{b}{2}\eta\frac{\partial^2 f}{\partial\eta^2}-c\frac{\partial^2 f}{\partial\eta\,\partial\xi}+h\left(\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2}-\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\,\partial\xi}\right)=0. \tag{3}$$

The coefficients a, b, c, d, e and h are defined by

$$a=\frac{g^{*2}}{u_\delta^*}\frac{\partial u_\delta^*}{\partial t^*},\quad b=\frac{\partial g^{*2}}{\partial t^*},\quad c=g^{*2}\frac{\partial\xi}{\partial t^*},$$

$$d=g^{*2}\frac{\partial u_\delta^*}{\partial x^*},\quad e=\tfrac{1}{2}u_\delta^*\frac{\partial g^{*2}}{\partial x^*},\quad h=u_\delta^*g^{*2}\frac{\partial\xi}{\partial x^*},$$

Case	$c(\xi)$	$h(\xi)$	Motion of singularity	Remarks
1	positive	positive	upstream	unsteady separation
2	positive	negative	downstream	Stewartson's singularity
3	negative	negative	upstream	—
4	negative	positive	downstream	—

TABLE 1

and, if semisimilar solutions are to exist, these coefficients must be functions of ξ alone. Additional details of the method of semisimilar solutions may be found in Williams & Johnson (1974). At this point it is convenient to write (3) as a system of two equations:

$$\frac{\partial^2 W}{\partial \eta^2}+\alpha_1\frac{\partial W}{\partial \eta}+\alpha_2 W+\alpha_3 = \alpha_4\frac{\partial W}{\partial \xi}, \tag{4}$$

$$\frac{\partial f}{\partial \eta} = W, \tag{5}$$

where

$$\alpha_1 = (d+e)f+h\frac{\partial f}{\partial \xi}+\tfrac{1}{2}b\eta, \quad \alpha_2 = -dW-a, \quad \alpha_3 = d+a, \quad \alpha_4 = c+hW.$$

This form, in which W is treated as one of the dependent variables, emphasizes the parabolic nature of (3). Furthermore, this is the form in which (3) is generally formulated for numerical solution. Stewartson (1951) has pointed out that in an equation that has the form of (4) a singularity may occur when the coefficient of a leading term in the ξ-derivative (the parabolic variable, i.e. α_4) of the differential equation vanishes. When α_4 changes sign in the interval of integration, the problem is referred to as singular parabolic.

It should be noted that (3) contains the two-dimensional steady boundary-layer formulation with the Goldstein singularity, as a special case for steady flow ($a(\xi) = b(\xi) = c(\xi) = 0$ and $\alpha_4 = h(\xi)\,\partial f/\partial\eta$). When α_4 vanishes just off the wall ($\partial f/\partial\eta = 0$) the Goldstein singularity occurs.

Suppose now that the coefficient α_4 vanishes (a singularity occurs) at some ξ-station, say ξ_0. The singularity occurs when $c(\xi_0) = -h(\xi_0)\,\partial f/\partial\eta$ and moves in the physical coordinate system with velocity

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{c(\xi_0)}{h(\xi_0)}u_\delta.$$

The direction of motion of the singularity is then determined by the signs of the coefficients $c(\xi)$ and $h(\xi)$. Table 1 indicates the various combinations of $c(\xi)$ and $h(\xi)$ and the associated singularities which have been studied.

In case 1 both $c(\xi)$ and $h(\xi)$ are positive, and the singularity occurs when $\partial f/\partial\eta$ becomes negative so that $c(\xi) = -h(\xi)\,\partial f/\partial\eta$. In a coordinate system moving with the singularity, as the singularity is approached, the flow field approaches one in which both the velocity and the shear vanish at a point away from the wall. These are just the characteristics postulated by Moore (1958), Rott (1956) and Sears (1956) for unsteady separation. These results are taken as verification of the Moore–Rott–Sears model for unsteady separation. It is noted, however, that results have only been obtained in case 1, where separation moves upstream. For case 4, which should correspond to separation moving downstream, no solutions have been obtained.

In case 2 the coefficient $c(\xi)$ is positive and the coefficient $h(\xi)$ is negative, and so α_4 approaches zero when the most negative value of $h(\xi)\,\partial f/\partial\eta$, which occurs at the outer edge of the boundary layer where $\partial f/\partial\eta = 1$, approaches the value of $c(\xi)$. In the case of a flat plate placed impulsively in motion, the shear on the plate varies according to Rayleigh's law ($\tau_w \approx t^{-\frac{1}{2}}$) until the effect of the leading edge is felt, and then undergoes a transition from the Rayleigh law to the Blasius law. That is, it takes a finite period of time for the effect of the leading edge to reach each point on the plate, and prior to that time the local flow behaves as if the plate were infinite. A similar argument may be made for flow past a wedge.

Case 3 represents an unrealistic (mathematically) case in which α_4 is never positive, so that (4) is never parabolic. Case 4 is, as indicated, a case that has not been studied as yet. This case is presently under investigation.

2.2. *Three-dimensional steady laminar boundary layers*

Let x and z be the orthogonal coordinates tangent to the body surface and y be the coordinate normal to the surface, with the corresponding velocity components u, w and v respectively. Let u_δ and w_δ be the x- and z-components of velocity at the outer edge of the boundary layer and ν be the kinetic viscosity. The boundary-layer equations for steady incompressible motion in three dimensions over a surface with large radii of curvature are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} = u_\delta\frac{\partial u_\delta}{\partial x}+w_\delta\frac{\partial u_\delta}{\partial z}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{7}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} = u_\delta\frac{\partial w_\delta}{\partial x}+w_\delta\frac{\partial w_\delta}{\partial z}+\nu\frac{\partial^2 w}{\partial y^2}. \tag{8}$$

The boundary conditions for this set of equations are

$$u(x,0,z) = v(x,0,z) = w(x,0,z) = 0,$$

$$\lim_{y\to\infty} u(x,y,z) = u_\delta(x,z), \quad \lim_{y\to\infty} w(x,y,z) = w_\delta(x,z).$$

Here again we reduce the number of independent variables from three to two by introducing new scaled coordinates η and ξ defined by

$$\eta = \frac{y(U/\nu l)^{\frac{1}{2}}}{g^*(x^*,z^*)}, \quad \xi = \xi(x^*,z^*)$$

and new dimensionless stream functions $F(\eta,\xi)$ and $G(\eta,\xi)$ defined such that

$$u = u_\delta\frac{\partial F}{\partial\eta}, \quad w = w_\delta\frac{\partial G}{\partial\eta}, \tag{9}$$

$$v = -\left(\frac{\nu U}{l}\right)^{\frac{1}{2}}\left(\frac{\partial u_\delta^* g^*}{\partial x^*}F+u_\delta^* g^*\frac{\partial\xi}{\partial x^*}\frac{\partial F}{\partial\xi}-u_\delta^*\frac{\partial g^*}{\partial x^*}\eta\frac{\partial F}{\partial\eta}\right.$$
$$\left.+\frac{\partial w^* g^*}{\partial z^*}G+w_\delta^* g^*\frac{\partial\xi}{\partial z^*}\frac{\partial G}{\partial\xi}-w_\delta^*\frac{\partial g^*}{\partial z^*}\eta\frac{\partial G}{\partial\eta}\right). \tag{10}$$

Case	$I(\xi)$	$H(\xi)$	Slope of singular line	Remarks
1	positive	positive	negative	three-dimensional separation
2	positive	negative	positive	—
3	negative	negative	negative	—
4	negative	positive	positive	—

TABLE 2

The continuity equation is satisfied identically by this choice of stream functions, and the x- and z-momentum equations become

$$\frac{\partial^3 F}{\partial \eta^3}+(A+B)F\frac{\partial^2 F}{\partial \eta^2}+(C+D)G\frac{\partial^2 F}{\partial \eta^2}+A\left(1-\left(\frac{\partial F}{\partial \eta}\right)^2\right)$$
$$+E\left(1-\frac{\partial G}{\partial \eta}\frac{\partial F}{\partial \eta}\right)+H\left(\frac{\partial^2 F}{\partial \eta^2}\frac{\partial F}{\partial \xi}-\frac{\partial F}{\partial \eta}\frac{\partial^2 F}{\partial \xi\,\partial \eta}\right)+I\left(\frac{\partial^2 F}{\partial \eta^2}\frac{\partial G}{\partial \xi}-\frac{\partial G}{\partial \eta}\frac{\partial^2 F}{\partial \xi\,\partial \eta}\right)=0, \quad (11)$$

$$\frac{\partial^3 G}{\partial \eta^3}+(C+D)G\frac{\partial^2 G}{\partial \eta^2}+(A+B)F\frac{\partial^2 G}{\partial \eta^2}+C\left(1-\left(\frac{\partial G}{\partial \eta}\right)^2\right)$$
$$+J\left(1-\frac{\partial F}{\partial \eta}\frac{\partial G}{\partial \eta}\right)+I\left(\frac{\partial^2 G}{\partial \eta^2}\frac{\partial G}{\partial \xi}-\frac{\partial G}{\partial \eta}\frac{\partial^2 G}{\partial \xi\,\partial \eta}\right)+H\left(\frac{\partial^2 G}{\partial \eta^2}\frac{\partial F}{\partial \xi}-\frac{\partial F}{\partial \eta}\frac{\partial^2 G}{\partial \eta\,\partial \xi}\right)=0. \quad (12)$$

In the transformed coordinate system the boundary conditions become

$$F(\xi,0)=G(\xi,0)=\frac{\partial F}{\partial \eta}(\xi,0)=\frac{\partial G}{\partial \eta}(\xi,0)=0,$$

$$\lim_{\eta\to\infty}\frac{\partial F}{\partial \eta}(\xi,\eta)=1, \quad \lim_{\eta\to\infty}\frac{\partial G}{\partial \eta}(\xi,\eta)=1.$$

The coefficients A, B, C, D, E, H, I and J are defined by

$$A=g^{*2}\frac{\partial u_\delta^*}{\partial x^*}, \quad B=\frac{u_\delta^*}{2}\frac{\partial g^{*2}}{\partial x^*}, \quad (13a,b)$$

$$C=g^{*2}\frac{\partial w_\delta^*}{\partial z}, \quad D=\frac{w_\delta^*}{2}\frac{\partial g^{*2}}{\partial z^*}, \quad (13c,d)$$

$$E=g^{*2}\frac{w_\delta^*}{u_\delta^*}\frac{\partial u_\delta^*}{\partial z^*}, \quad H=g^{*2}u_\delta^*\frac{\partial \xi}{\partial x^*}, \quad (13e,f)$$

$$I=g^{*2}w_\delta^*\frac{\partial \xi}{\partial z^*}, \quad J=g^{*2}\frac{u_\delta^*}{w_\delta^*}\frac{\partial w_\delta^*}{\partial x^*}, \quad (13g,h)$$

and, if semisimilar solutions are to exist, these coefficients must be functions of ξ alone. Here the velocity components, coordinates and g scaling factor have been non-dimensionalized according to $u_\delta^*=u/U$, $w_\delta^*=w_\delta/U$, $g^*=(U/l)^{\frac{1}{2}}g$, $x^*=x/l$, $z^*=z/l$, where U and l are some characteristic velocity and length for the problem under consideration. Additional details of the method of semisimilar solutions applied to three-dimensional steady boundary layers may be found in Williams (1975).

It should be noted that (11) and (12) contain the steady two-dimensional problem as a special case in which $w_\delta^*=0$ and $\partial/\partial z=0$ (hence $C=E=I=D=J=0$ and

$G(\xi,\eta) = 0$). In addition, (11) and (12) contain as a special case the flows over infinite swept cylinders, in which $w_\delta^* = w_\delta^*(x^*)$ and $\partial/\partial z^* = 0$ (hence $C = E = I = D = J = 0$).

As in the unsteady case, it is convenient to rewrite (11) and (12) as the system of equations

$$\frac{\partial^2 W_1}{\partial\eta^2} + \alpha_{11}\frac{\partial W_1}{\partial\eta} + \alpha_{12} W_1 + \alpha_{13} = \alpha_{14}\frac{\partial W_1}{\partial\xi}, \tag{14}$$

$$\frac{\partial^2 W_2}{\partial\eta^2} + \alpha_{21}\frac{\partial W_2}{\partial\eta} + \alpha_{22} W_2 + \alpha_{23} = \alpha_{24}\frac{\partial W_2}{\partial\xi}, \tag{15}$$

$$W_1 = \frac{\partial F}{\partial\eta}, \quad W_2 = \frac{\partial G}{\partial\eta}, \tag{16}$$

where

$$\alpha_{11} = \alpha_{21} = \left[(A+B)F + (C+D)G + H\frac{\partial F}{\partial\xi} + I\frac{\partial G}{\partial\xi}\right],$$

$$\alpha_{12} = -\left[A\frac{\partial F}{\partial\eta} + E\frac{\partial G}{\partial\eta}\right], \quad \alpha_{22} = -\left(J\frac{\partial F}{\partial\eta} + C\frac{\partial G}{\partial\eta}\right),$$

$$\alpha_{13} = A+E, \quad \alpha_{23} = C+J,$$

$$\alpha_{14} = \alpha_{24} = \left[H\frac{\partial F}{\partial\eta} + I\frac{\partial G}{\partial\eta}\right].$$

We note here that, as in the unsteady case, the formulation of the problem as in (14)–(16) emphasizes the parabolic nature of the problem (so long as $\alpha_{14} = \alpha_{24}$ is positive), and is the form in which (11) and (12) are often formulated for numerical solution. Further, we can see that, as in the unsteady case, one would expect a singularity in the solution as α_{14} (or α_{24}) approaches zero.

Separation of three-dimensional incompressible laminar boundary layers was studied using the technique of semisimilar solutions in Williams (1975). Although the nature of three-dimensional separation was determined there, the direct correlation between separation and the vanishing of the coefficient α_{14} (or α_{24}) was not made. For the sake of completeness, the direct correlation between three-dimensional steady separation and the vanishing of α_{14} will be demonstrated later. It suffices at this point to note that such a correlation does exist.

Assuming that such a correlation exists, then separation occurs at the point (line) where $\alpha_{14} = 0$, or when $I(\xi)\,\partial G/\partial\eta = -H(\xi)\,\partial F/\partial\eta$ and the slope of the singular line is (at $\xi_0 = \xi$) given by

$$\left.\frac{\mathrm{d}z^*}{\mathrm{d}x^*}\right|_{\xi_0} = -\frac{H(\xi)}{I(\xi)}\frac{w_\delta^*}{u_\delta^*}.$$

Table 2 indicates the various combinations of the coefficients $I(\xi)$ and $H(\xi)$ and the associated singularity that has been studied.

In case 1 both $I(\xi)$ and $H(\xi)$ are positive, so that α_{14} becomes zero when either $\partial F/\partial\eta$ or $\partial G/\partial\eta$ become sufficiently negative that $H(\xi)\,\partial F/\partial\eta = -I(\xi)\,\partial G/\partial\eta$. As the singularity is approached, the flow may exhibit one of two possible behaviours (Williams 1975). In the first of these, both the x- and z-components of shear approach zero simultaneously, so that the total shear is zero along the separation line. This type of separation has been termed by Maskell (1955) 'singular separation'. In the second case the 'limiting' or 'wall' streamlines turn so that they all become tangent to a singular line at separation; i.e. the separation line is an envelope of the 'limiting'

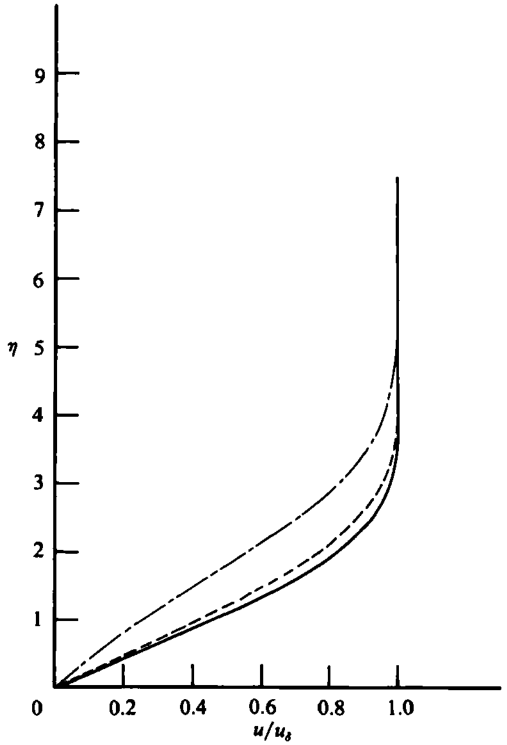

FIGURE 1. Velocity profiles for the x-component of velocity; $\xi = x^* + \beta z^*$, $u_\delta^* = 1$, $w_\delta^* = 1 - \xi$, $\beta = 1$: ——, $\xi = 0$; ------, 0.1; —·—, 0.239.

streamlines. This type of separation has been termed by Maskell (1955) ordinary separation. Case 2 has apparently never been studied, but the similarity in form of (14)–(16) with (4) and (5) for two-dimensional unsteady flow and the similarities between tables 1 and 2 suggests a strong similarity between this case and case 2 of table 1. Case 3 represents an unrealistic case (mathematically) in which α_{14} is always negative so that (14) and (15) are never parabolic. Case 4 is a case that remains to be studied.

We have noted the similarity between the equations of motion for the two-dimensional unsteady and the three-dimensional cases when formulated for semisimilar solutions. This similarity is particularly evident when (4) and (5) for two-dimensional unsteady flow are compared with (14)–(16) for steady three-dimensional flow. In light of this, it is intriguing to note that, while solutions have been obtained for two separate types of singularity associated with two-dimensional unsteady flow (see table 1), solutions have only been obtained for one type of singularity associated with three-dimensional steady flow (see table 2). For two-dimensional unsteady-flow solutions with singularities have been obtained in case 1 ($c(\xi) > 0$, $h(\xi) > 0$) and case 2 ($c(\xi) > 0$, $h(\xi) < 0$), while for three-dimensional steady flows solutions with singularities have only been obtained for case 1 ($H(\xi) > 0$, $I(\xi) > 0$). Quite naturally, the question arises as to whether or not there are solutions with singularities possible for case 2 for three-dimensional steady flows; is there a counterpart to the Stewartson singularity in the three-dimensional case? We shall pursue that question further later and demonstrate that a new singularity exists in solutions to the three-dimensional steady laminar-boundary-layer equations for case 2 noted above and that this

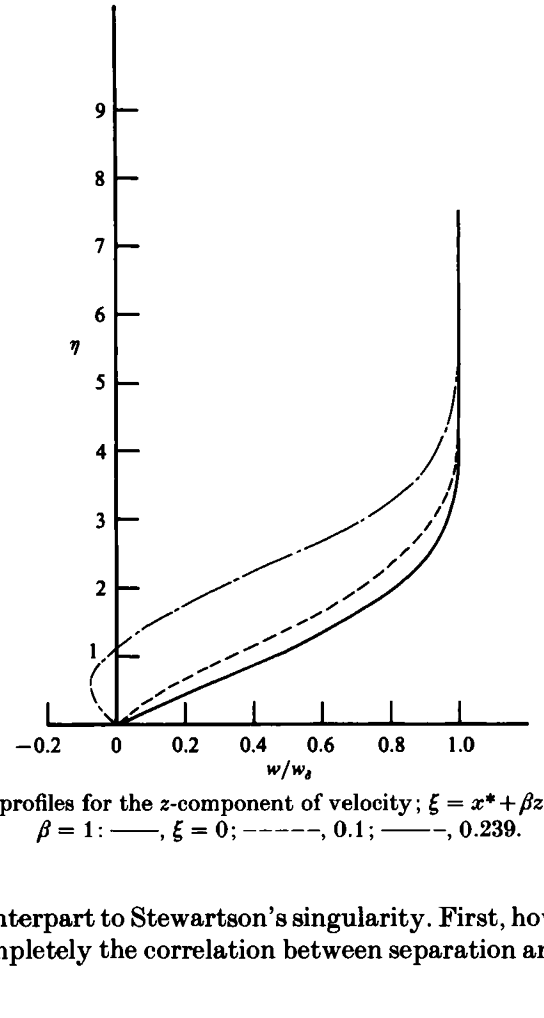

FIGURE 2. Velocity profiles for the z-component of velocity; $\xi = x^* + \beta z^*$, $u_\delta^* = 1$, $w_\delta^* = 1 - \xi$, $\beta = 1$: ——, $\xi = 0$; – – – – –, 0.1; —·—·—, 0.239.

singularity is a counterpart to Stewartson's singularity. First, however, it is necessary to demonstrate completely the correlation between separation and the coefficient α_{14}.

3. Three-dimensional steady separation reviewed

As mentioned earlier, three-dimensional steady laminar boundary layers were studied using the technique of semisimilar solutions (Williams 1975). At the time of that work, however, the significance of the vanishing of the coefficient α_{14} was not recognized. Although the nature of three-dimensional laminar separation was noted, the correlation between separation and the vanishing of α_{14} was not made. To make that correlation clearly, we now study an entirely new three-dimensional laminar boundary layer that approaches separation, employing the technique of semisimilar solutions. We consider the three-dimensional steady boundary-layer problem in which

$$\xi = x^* + \beta z^*,$$

$$u_\delta^* = 1, \quad w_\delta^* = 1 - \xi = 1 - x^* - \beta z^*,$$

$$g^{*2} = \frac{\xi}{u_\delta^*} = \xi.$$

This is a very simple example of a three-dimensional flow for which a semisimilar solution exists. The x-component of the external velocity is uniform, while the

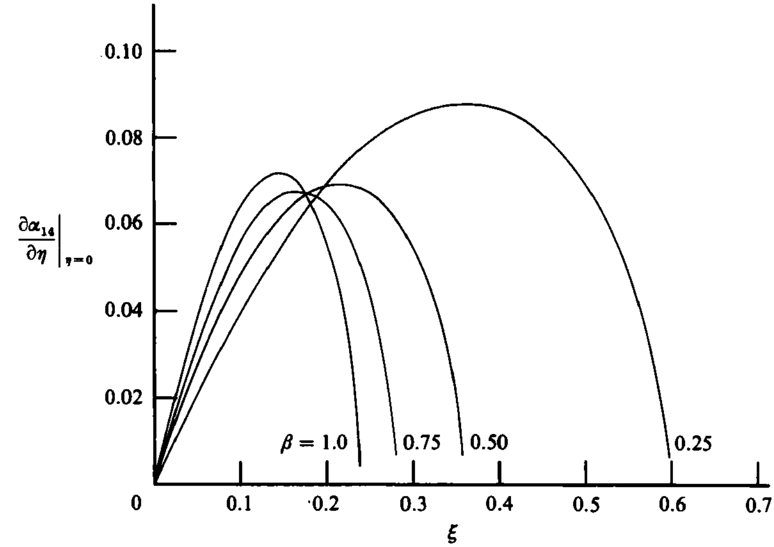

FIGURE 3. Variation of $\partial\alpha_{14}/\partial\eta\,|_{\eta=0}$ with $\xi = x^*+\beta z^*$; $u_\delta^* = 1$, $w_\delta^* = 1-\xi$.

z-component is linearly retarded in both the x- and z-directions. The pressure gradient in the x-direction is zero, while that in the z-direction is given by

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{U_\infty^2}{l}\{(1+\beta)-\beta x^*-\beta^2 z^*\}.$$

The pressure gradient is adverse, but its magnitude decreases as either x^* or z^* increases. This is a three-dimensional counterpart to the Howarth's classical two-dimensional linearly retarded flow.

As in the cases studied in Williams (1975), the solution is begun at $\xi = 0$, where a similar solution exists, and is marched forward in the direction of increasing ξ. At each ξ-station an iteration is required as a result of the nonlinearity of the problem. At some downstream station the number of iterations required to obtain a converged solution starts to grow with each succeeding station, until at one station convergence cannot be obtained in a reasonable number of iterations. This behaviour is taken, by analogy with finite-difference calculation of the two-dimensional laminar boundary layer, as an indication of approaching a point of singular behaviour in the solution of the laminar-boundary-layer equations. It is possible to track the variation of the coefficient α_{14} as the solution proceeds. As noted earlier, this coefficient is given by

$$\alpha_{14} = H(\xi)\frac{\partial F}{\partial\eta}+I(\xi)\frac{\partial G}{\partial\eta}.$$

For the present case $H(\xi) = \xi$ and $I(\xi) = \beta\xi(1-\xi)$. Thus the coefficient α_{14} can only vanish if either $\partial F/\partial\eta$ or $\partial G/\partial\eta$ becomes negative. This is indeed what happens. Figures 1 and 2 show velocity profiles ($\partial F/\partial\eta$ and $\partial G/\partial\eta$ respectively) at several values of ξ for the case in which $\beta = 1$. Beyond $\xi \approx 0.180$ the w velocity component is negative near the wall. The coefficient α_{14} is always zero at the wall. The minimum value of α_{14} in the fluid occurs immediately adjacent to the wall. The vanishing of α_{14} in the fluid (i.e. not on the wall) occurs when $\partial\alpha_{14}/\partial\eta\,|_{\eta=0} = 0$. Thus, instead of tracking α_{14} at some arbitrary point near the wall to determine when this coefficient

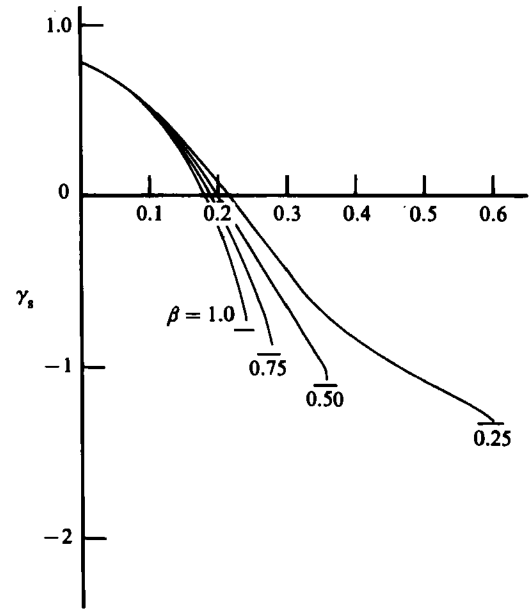

FIGURE 4. Variation of wall streamline angle γ_s with $\xi = x^* + \beta^*$; $u_\delta^* = 1$, $w_\delta^* = 1 - \xi$. Limiting values of γ_ξ corresponding to the limit value of ξ are indicated by short horizontal lines.

vanishes in the fluid, we have tracked $\partial\alpha_{14}/\partial\eta\,|_{\eta=0}$. The variation of this derivative with ξ is shown in figure 3 for $\beta = 0.25$, 0.50, 0.75 and 1.0. As ξ increases from zero, $\partial\alpha_{14}/\partial\eta\,|_{\eta=0}$ increases, reaches a peak and then drops precipitously toward zero. As $\partial\alpha_{14}/\partial\eta\,|_{\eta=0}$ decreases toward zero the number of iterations required for convergence at each station increases, indicating, as noted above, the approaching of a singularity. For the case in which $\beta = 1.0$, extrapolation of the results indicates that the singularity occurs at $\xi \approx 0.24$.

The question now arises as to what, if any, physical or mathematical significance can be associated with the singularity. We note that the angle γ_s of the limiting streamlines (or wall streamlines), the projection of the streamlines closest to the wall, on the wall, is given by

$$\tan\gamma_s = \lim_{\eta\to 0} \frac{w_\delta\,\partial G/\partial\eta}{u_\delta\,\partial F/\partial\eta} = \frac{w_\delta\,\partial^2 G/\partial\eta^2\,|_{\eta=0}}{u_\delta\,\partial^2 F/\partial\eta^2\,|_{\eta=0}}.$$

The angle for the lines of constant ξ, including the one corresponding to the singularity, is given by

$$\gamma_\xi = \arctan\frac{dz}{dx}\bigg|_{\xi=\text{const}} = \arctan\left(-\frac{1}{\beta}\right).$$

The angle γ_s of the wall streamlines is presented in figure 4 as a function of ξ for each of the values of β studied. In all cases the angle of the wall streamlines at $\xi = 0$ is 45° (0.785 rad), as one might expect. As ξ increases, γ_s decreases, approaching a limiting value as the separation singularity is approached. We note that the limiting value of γ_s approached in each case is just the value of γ_ξ for that case (value of β). The value of γ_ξ for each case is denoted by a short horizontal line in figure 4. Physically, this indicates that for any given case the limiting streamlines are approaching a limiting line along which the limiting streamlines are becoming tangent

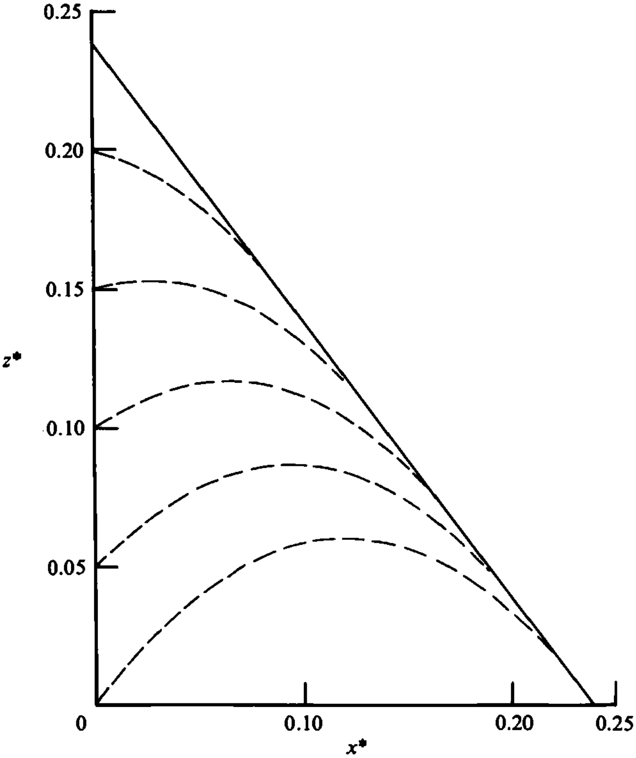

FIGURE 5. Limiting streamlines and line of constant ξ corresponding to singularity; $\xi = x^* + \beta z^*$, $u_\delta^* = 1$, $w_\delta^* = 1-\xi$, $\beta = 1$: -----, streamlines; ——, $\xi = 0.239$.

to the line of separation. This type of separation is termed 'ordinary' separation by Maskell (1955). This result is verified in figure 5, where several streamlines have been plotted in the (x^*, z^*)-plane for the case $\beta = 1$. Also plotted is the limiting value of ξ corresponding to the singularity; i.e. $\xi_0 \approx 0.24$.

We note that in the case of three-dimensional separation the vertical component of velocity near the wall increases rapidly as separation is approached. Figure 6 shows the variation of the calculated vertical component of velocity v^* near the wall (at $\eta = 0.02$) for the case $\beta = 1.0$. The rapid increase of v^* near separation is quite evident here. We note that this behavior is similar to the rapid increase in the vertical component of velocity near separation in unsteady two-dimensional boundary-layer solutions (Williams & Johnson 1974).

Finally, it is interesting to note that along the line of singularity that corresponds to separation both u_δ^* and w_δ^* are constant, so that along this line the total velocity and static pressure in the external stream are constant. This is consistent with an observation made by Telionis & Costis (1983) based on an experimental study of three-dimensional laminar separation.

4. A new singularity

We now consider the question posed earlier: is there a three-dimensional counterpart to the Stewartson singularity found in certain two-dimensional unsteady flows? If such a singularity does occur it should have the same general properties as the

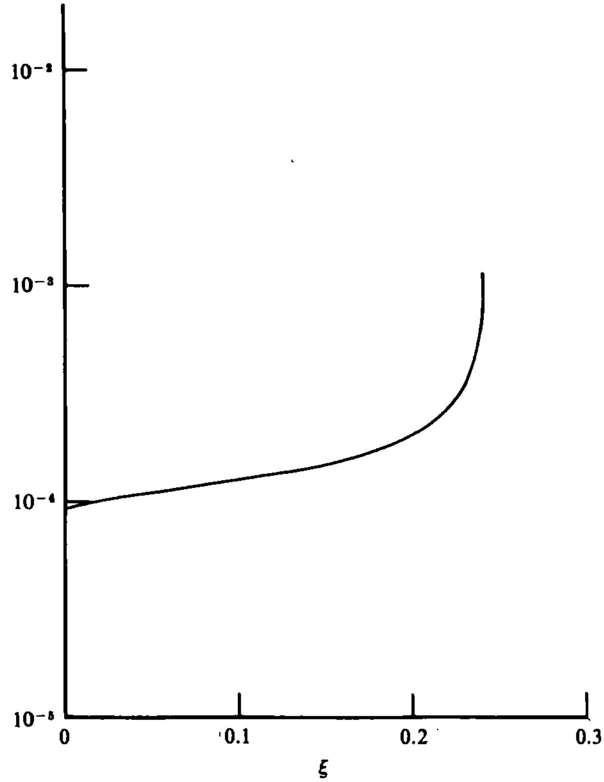

FIGURE 6. Variation of vertical component of velocity at $\eta = 0.02$; $\xi = x^*+\beta z^*$, $u^*_\delta = 1$, $w^*_\delta = 1-\xi$, $\beta = 1$.

Stewartson singularity. That is, (i) it should occur in a favourable or zero pressure gradient, and (ii) it should occur at the outer edge of the boundary layer. Using these two properties as a guide, we have generated a family of flows in which this singularity occurs. We now study this family of flows in detail to determine first the general nature of the singularity and secondly the physical and mathematical nature of the flow approaching the singularity.

The family of flows to be considered is a simple three-dimensional counterpart of the classical two-dimensional Falkner–Skan flows, in which

$$u^*_\delta = 1, \quad w^*_\delta = z^{*\beta},$$

where β is a constant. Guided by previous work, we take the scaling function g^{*2} to be x^*/u^*_δ. It is now necessary to determine the appropriate scaling function ξ. Along lines of constant ξ we have

$$d\xi = 0 = \frac{\partial \xi}{\partial x^*}dx^* + \frac{\partial \xi}{\partial z^*}dz^*,$$

or, using (13*f*, *g*),

$$\left.\frac{dz^*}{dx^*}\right|_\xi = \frac{\partial\xi/\partial x^*}{\partial\xi/\partial z^*} = -\frac{H(\xi)}{I(\xi)}\frac{w^*_\delta}{u^*_\delta} = -\frac{H(\xi)}{I(\xi)}z^{*\beta}.$$

Integration leads us to the conclusion that the scaling function ξ must be some function of the combination $x^*/z^{*(1-\beta)}$. We choose here the simplest of the functional

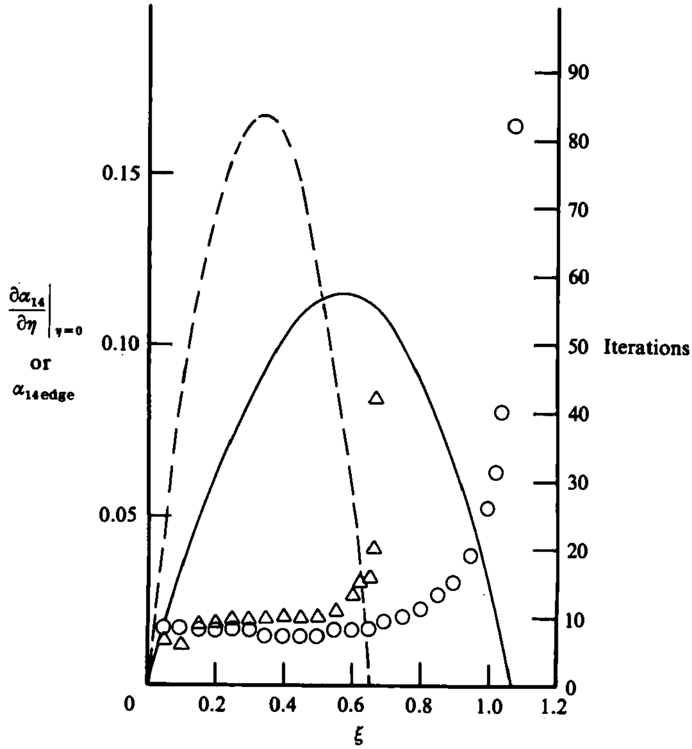

FIGURE 7. Variation of α_{14}, $\partial\alpha_{14}/\partial\eta|_{\eta=0}$ and the number of iterations required for convergence with ξ: ——, $\partial\alpha_{14}/\partial\eta|_{\eta=0}$; ○, number of iterations for convergence for $\beta = 0.5$; -----, $\alpha_{14\text{edge}}$; △, number of iterations for convergence for $\beta = -0.5$.

forms available; i.e. $\xi = x^*/z^{*(1-\beta)}$. With these choices of u_δ^*, w_δ^*, g^{*2} and ξ, the coefficients (13a–h) that appear in (11) and (12) become

$$A(\xi) = 0, \quad B(\xi) = \tfrac{1}{2},$$
$$C(\xi) = \beta\xi, \quad D(\xi) = 0,$$
$$E(\xi) = 0, \quad H(\xi) = \xi,$$
$$I(\xi) = -(1-\beta)\,\xi^2, \quad J(\xi) = 0.$$

With all of these coefficients now defined, solution of (11) and (12) is straightforward.

For this simple flow the pressure gradient in the x-direction is zero, while the pressure gradient in the z-direction is given by

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{U_\infty^2}{l}\{\beta z^{*2\beta-1}\}.$$

The pressure gradient in the z-direction is favourable (negative) for positive values of β and unfavourable (positive) for negative values of β. In the spirit noted above, then, solutions were initially obtained for positive values of β of 0, 0.2, 0.5 and 0.8. The results obtained in these cases were so different from what was expected that the range of β studied was extended to cover a range of negative β of -0.5, -1.0, -1.5, -2.0.

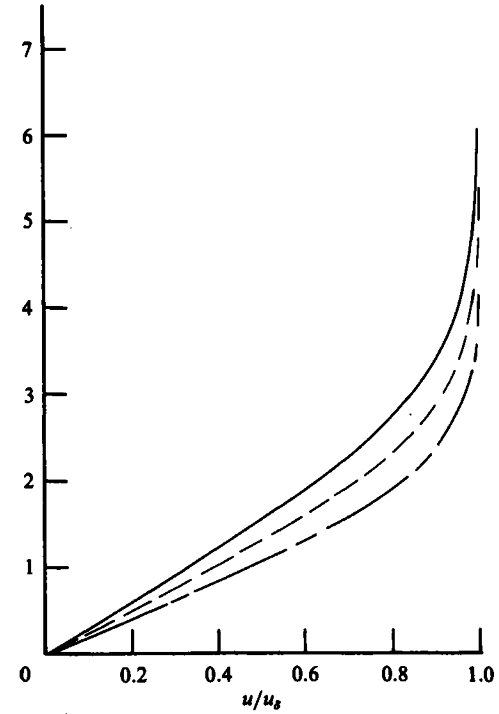

FIGURE 8. Velocity profile for the x-component of velocity; $\xi = x^*/z^{*(1-\beta)}$, $u_s^* = 1$, $w_s^* = z^{*\beta}$, $\beta = 0.5$: ——, $\xi = 0$; - - - - -, 0.6; — — —, 1.07.

4.1. *Favourable pressure gradient*

As in the cases where separation is encountered (§3), the solutions start at $\xi = 0$, where a similar solution exists, and proceed in the direction of increasing ξ. At some downstream station the number of iterations required for convergence begins to grow with each succeeding station, until at one station convergence cannot be obtained in a reasonable number of iterations (120). Again, this behaviour is taken, by analogy with the finite-difference calculation of the two-dimensional steady, two-dimensional unsteady, and three-dimensional steady boundary layer, as an indication of approaching a point (line) of singular behaviour in the solution of the laminar-boundary-layer equations. At the same time as the number of iterations required for convergence is tracked, the value of α_{14} at the outer edge of the boundary layer is also tracked, for it was anticipated that the singularity would be related to the vanishing of this coefficient. Surprisingly, the singularity was *not* associated with the vanishing of α_4 at the outer edge of the boundary layer but with the vanishing of this coefficient near the wall. Here again α_{14} is always zero at the wall and the condition we are looking for is the vanishing of the coefficient α_{14} in the fluid near the wall. This occurs when $\partial\alpha_{14}/\partial\eta\,|_{\eta=0} = 0$. Thus for these flows we have tracked $\partial\alpha_{14}/\partial\eta\,|_{\eta=0}$ to determine when α vanishes in the fluid. Figure 7 shows the variation of $\partial\alpha_{14}/\partial\eta\,|_{\eta=0}$ for the case $\beta = +0.5$ from $\xi = 0$ to $\xi = 1.07$, the last station at which convergence can be obtained in less than 120 iterations. As ξ increases from zero $\partial\alpha_{14}/\partial\eta\,|_{\eta=0}$ increases, reaches a maximum and then drops precipitously to zero. (In fact, at the last station at which convergence was obtained in a reasonable number

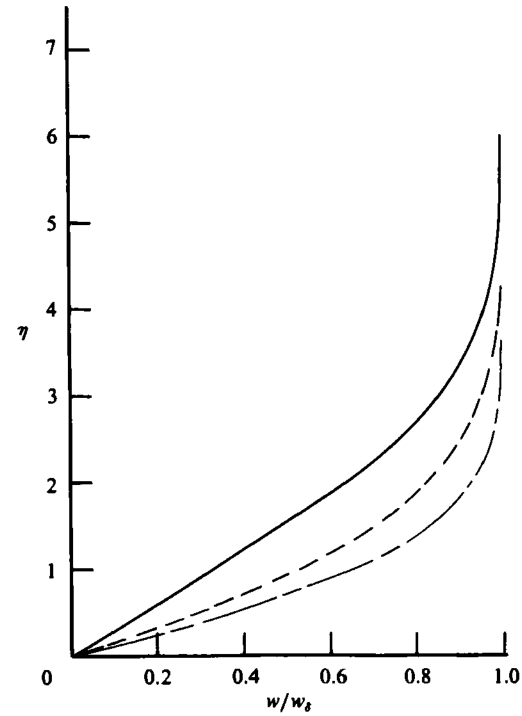

FIGURE 9. Velocity profiles for z-component of velocity; $\xi = x^*/z^{*(1-\beta)}$, $u_\delta^* = 1$; $w_\delta^* = z^{*\beta}$, $\beta = 0.5$: ——, $\xi = 0$; - - - - -, 0.6; ——-, 1.07.

of iterations, $\partial\alpha_{14}/\partial\eta\,|_{\eta=0}$ was very slightly negative.) Also shown in figure 7 is the number of iterations required for convergence at each station.

For $\beta = 0.5$ we have $u_\delta^* = 1$, $w_\delta^* = z^{*\frac{1}{2}}$, $H(\xi) = \xi$ and $I(\xi) = -0.5\xi^2$. Thus the coefficient α_{14} can vanish near the wall with both $\partial F/\partial\eta$ and $\partial G/\partial\eta$ positive. This is indeed what happens. Figures 8 and 9 show both the u- and w-velocity profiles at several ξ-stations for the case $\beta = 0.5$. These velocity profiles are typical of the results obtained for other values of β for $\beta > 0$. The only peculiarity in the profiles that might indicate what might be happening is the rather obvious fact that the boundary layer is becoming thinner as the singularity is approached! This is consistent with behaviour of the boundary layer on a plate impulsively placed into motion (Stewartson's singularity) and just the opposite of the boundary-layer behaviour as three-dimensional separation is approached.

4.2. *Adverse pressure gradient*

As mentioned earlier, there initially had been no intention of obtained solutions for negative values of β, i.e. for an adverse pressure gradient. It had been anticipated that solutions for flows with an adverse pressure gradient would simply yield separating flows, and thus would provide no new information beyond that provided in §3. When consideration of positive values of β failed to yield a singularity at the outer edge of the boundary layer, the range of β considered was extended to include negative values of β, as noted above.

As in the earlier cases, these solutions start at $\xi = 0$, where a similar solution exists, proceed in the direction of increasing ξ. Again, at some downstream station the

β	ξ_c	Last value of ξ for convergence
0.5	$\frac{2}{3}$	0.665
−1.0	$\frac{1}{2}$	0.495
−1.5	$\frac{2}{5}$	0.39
−2.0	$\frac{1}{3}$	0.33

TABLE 3. Comparison of calculated values of ξ_c with the last value of ξ for which convergence could be obtained

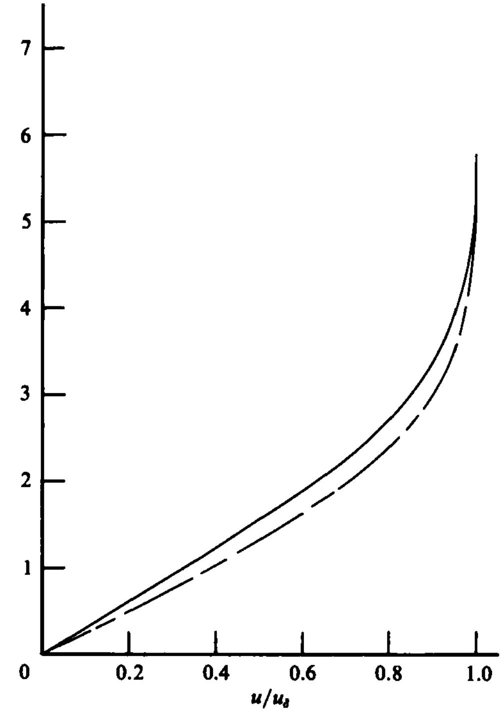

FIGURE 10. Velocity profiles for the x_β-component of velocity; $\xi = x^*/z^{*(1-\beta)}$, $u_\delta^* = 1$, $w_\delta^* = z^{*\beta}$, $\beta = -0.5$; ——, $\xi = 0$; – – –, 0.665.

number of iterations required for convergence begins to grow with each succeeding station until one-station convergence cannot be obtained in a reasonable number of iterations. Again this behaviour is taken, as explained above, as an indication of approaching a point (line) of singular behaviour in the solution of the laminar-boundary-layer equations. At the same time as the number of iterations required for convergence is tracked, the value of α_{14} at the outer edge of the boundary layer is also tracked. In this case it was found that the approaching of the singularity was indeed associated with the vanishing of the coefficient α_{14} at the edge of the boundary layer. Figure 7 shows the variation of α_{14} at the outer edge of the boundary layer and the number of iterations required for convergence for the case $\beta = -0.5$ from $\xi = 0$ to $\xi = 0.665$, the last station at which convergence can be obtained in less than 120 iterations. As ξ increases, α_{14} at the outer edge increases, reaches a maximum, and then drops precipitously towards zero; at the same time the number of iterations required for convergence increases sharply.

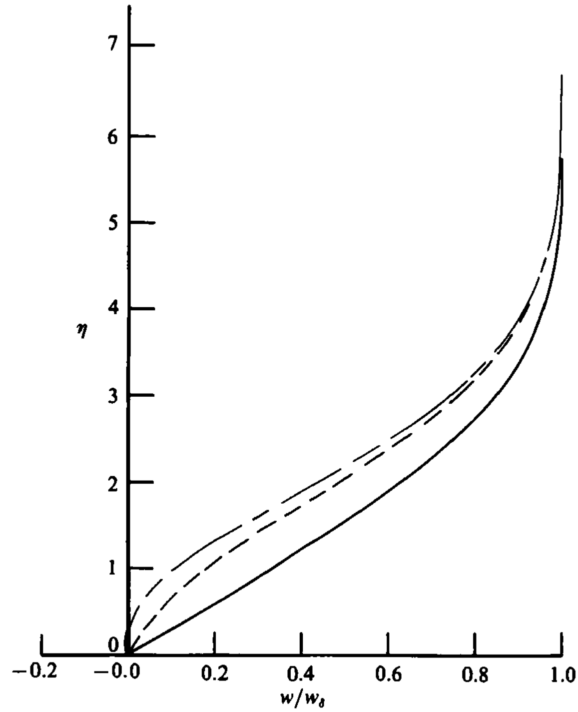

FIGURE 11. Velocity profiles for the z-component of velocity; $\xi = x^*/z^{*(1-\beta)}$, $u_\delta = 1$, $w_\delta^* = z^{*\beta}$, $\beta = -0.5$; ——, $\xi = 0$; - - - - -, 0.4; — — —, 0.665.

For $\beta = -0.5$ we have $u_\delta^* = 1$, $\omega_\delta^* = z^{*-\frac{1}{2}}$, $H(\xi) = \xi$, $I(\xi) = -1.5\xi^2$. The coefficient α_{14} can vanish at the outer edge of the boundary layer with $\partial F/\partial\eta$ and $\partial G/\partial\eta$ both positive. In fact, since both $\partial F/\partial\eta$ and $\partial G/\partial\eta$ are unity at the outer edge, α_{14} at the outer edge is given by $\alpha_{14\text{edge}} = -(1-\beta)\xi^2+\xi$. Thus the value of ξ at which $\alpha_{14\text{edge}}$ vanishes is given by $\xi_c = 1/(1-\beta)$. Table 3 compares the value of ξ_c with the last value of ξ at which convergence can be obtained, for several values of β. Clearly, the last values of ξ for which convergence was obtained are close to the predicted values of ξ_c for which $\alpha_{14\text{edge}}$ vanished.

With an adverse pressure gradient in the z-direction it was expected that the z-component of shear would decrease and perhaps even become negative. This is indeed what happens, as shown in figures 10 and 11. As indicated in figure 10, the u-component of velocity is little affected by changes in ξ. The velocity profile for $\xi = 0.4$ is, for all practical purposes, the same as that for $\xi = 0$, and is thus not plotted in figure 10. The w-component of velocity, however, varies strongly with ξ, as shown in figure 11. The adverse pressure gradient in the z-direction causes a strong reduction in shear (with increasing ξ) and, close to the singularity, a slight backflow in the z-direction. If should be noted, however, that the pressure gradient, in all cases studied, was never sufficiently strong to promote separation before the outer-edge singularity was encountered.

4.3. *The significance of the singularity*

A review of the semisimilar formulation of the two-dimensional unsteady and the three-dimensional steady boundary-layer equations indicated that there might be a

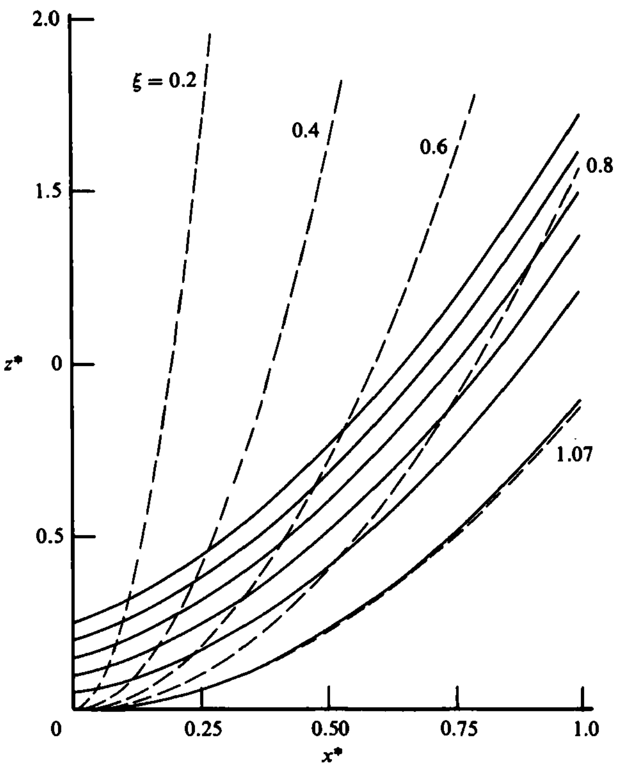

FIGURE 12. Wall streamlines and lines of constant $\xi = x^*/z^{*(1-\beta)}$; $u_\delta^* = 1$, $w_\delta^* = z^{*\beta}$, $\beta = 0.5$: ——, wall streamlines; -----, line of constant ξ.

new type of singularity in the solutions to the three-dimensional steady laminar-boundary-layer equations. Numerical solutions of the three-dimensional steady boundary-layer equations in the semisimilar formulation has indicated not one but two new singularities. The question that now arises is: 'what is the significance of these singularities?'

The fact that the singularities occur either at the outer edge of the boundary layer or at the wall suggests that we first look in these areas. A great deal can be learned by tracking the angle of the outer-edge streamlines or the wall streamlines as in the case of separation (§3). The numerical results indicate that in the case of favourable pressure gradients ($\beta > 0$) the slope of the wall streamlines approaches the slope of the constant-ξ line corresponding to the singularity as the singularity is approached. This is exactly the behaviour encountered in the case where separation is approached (figure 4). In the case of adverse pressure gradients ($\beta < 0$) the slope of the external streamlines (streamlines at the outer edge of the boundary layer) approach the slope of the constant-ξ line corresponding to the singularity as the singularity is approached. These results could be shown graphically, as in figure 4. It is more instructive, however, not to look at the streamline angle but at the streamlines and the lines of constant ξ. Figure 12 shows several wall streamlines as well as several lines of constant ξ, for a case with a favourable pressure gradient ($\beta = 0.5$), and figure 13 shows several external streamlines as well as several lines of constant ξ for a case with an adverse pressure gradient ($\beta = -0.5$). In each case the last value of ξ at which a solution can

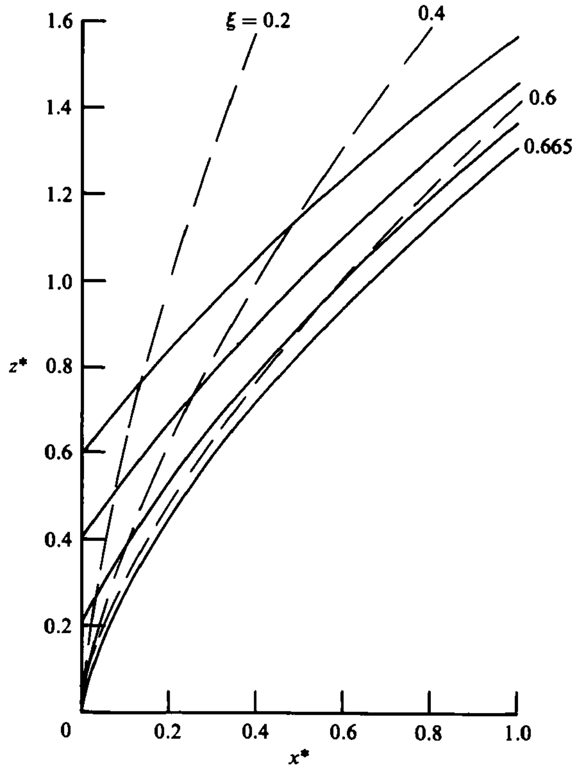

FIGURE 13. Outer-edge streamlines and lines of constant $\xi = x^*/z^{*(1-\beta)}$; $u_\delta^* = 1$, $w_\delta^* = z^{*\beta}$, $\beta = -0.5$; ——, outer-edge streamlines; -----, lines of constant ξ.

be obtained corresponds very closely to the streamline (the wall streamline for $\beta = 0.5$, the external streamline for $\beta = -0.5$) that passes through the origin ($x^* = z^* = 0$). The singularity occurs at the value of ξ that corresponds to the streamline which defines the limit of the domain of influence for the given set of initial ($\xi = 0$) conditions. This is simply an application of the Raetz 'influence principle' (Raetz 1957; Nash & Patel 1972), which defines domains of influence and domains of dependence for three-dimensional boundary-layer calculations. This principle is based on the fact that a disturbance at a point in the boundary layer is instantly transmitted up and down the normal to the body surface through that point and is convected downstream along all the streamlines passing through this line. Thus for each point on the body there is a zone of influence and a zone of dependence, in the shape of curvilinear wedges, one opening in the upstream direction and the other in the downstream direction. The zones of influence and dependence are bounded by the streamlines of maximum and minimum angles passing through the body normal at the point in question. A clear and succinct discussion of the Raetz influence principle including its implications with respect to upstream or initial conditions is given by Nash & Patel (1972).

Consider now the wall streamlines and lines of constant ξ shown in figure 12 for $\beta = 0.5$. The wall streamlines provide only one of the boundaries of regions of influence and dependence. The other boundary is given by the streamlines at the outer edge of the boundary layer. These external streamlines are omitted in figure 12 in order to avoid cluttering the figure. We note, however, that the slope of the outer-edge streamline is given by

$$\left.\frac{dz^*}{dx^*}\right|_{\text{edge}} = \frac{w_\delta^*}{u_\delta^*},$$

while the slope of the wall (limiting) streamlines is given by

$$\left.\frac{dz^*}{dx^*}\right|_{\text{wall}} = \frac{w_\delta^*}{u_\delta^*}\frac{G''(\xi,0)}{F''(\xi,0)}.$$

For $\beta = 0.5$ (in fact for all cases with a favourable pressure gradient) $G''(\xi,0)/F''(\xi,0)$ is greater than unity. Thus at a given point the slope of outer-edge streamline will be smaller than the slope of the wall streamline. In particular, the external streamline through the origin will lie below the wall streamline through the origin.

Now any point above the wall streamline through the origin (or for $\xi < 1.07$) lies in the domain of influence of the points along the line $\xi = 0$ ($x^* = 0$), and hence can be calculated since the initial conditions along this line are given. On the other hand, any point below the streamline through the origin (and hence for $\xi > 1.07$) lies in the domain of influence of points along the line $z^* = 0$, $x^* > 0$ ($\xi \to \infty$). Points below $\xi = 1.07$ therefore cannot be calculated because they do not lie in the domain of influence of the given initial conditions (along $\xi = 0$). The singularity occurs along and the calculation is limited by the boundary of the domain of influence of the initial conditions for the flow.

A similar argument may be made for the case where $\beta = -0.5$ (and in fact for the cases studied for which $\beta < 0$). For these cases $G''(\xi,0)/F''(\xi,0)$ is always less than zero, so that the wall streamline at a given point always has a smaller slope than the external streamline through that point. In particular, the wall (or limiting) streamline though the origin has a smaller slope than, and thus lies below, the external streamline through this position. In this case, then, it is the external streamline which determines the limiting extent of the calculation. The calculation can be continued up to $\xi = 0.665$ which corresponds to the external streamline through the origin. Above this streamline all points lie in the domain of influence of the initial conditions along $\xi = 0$. Points beyond $\xi = 0.665$, i.e. below the streamline through the origin, (and $\xi = 0.665$) do not lie in the domain of influence for the given initial ($\xi = 0$) conditions, but lie instead in the domain of influence of points along the line $z^* = 0$, $x^* > 0$ ($\xi \to \infty$). Again, points beyond $\xi = 0.665$ cannot be calculated in the present coordinate system since they are influenced by the 'downstream' conditions at $\xi = \infty$.

It is concluded, then, that the new singularities that have been observed here correspond to the limits of integration of the three-dimensional laminar-boundary-layer equations as dictated by the Raetz influence principle. The singularity occurs along the line that corresponds to the streamline marking the boundary of the region where the flow is influenced by the given set of initial conditions.

It is not surprising that these singularities in the solution of the three-dimensional laminar-boundary-layer equations are analogous to the Stewartson singularity in unsteady two-dimensional flow. The Stewartson singularity marks the boundary between the region that is uninfluenced by the leading edge ($Ut/x < 1$) and the region that is influenced by the leading edge ($Ut/x > 1$). In the case of the singularities studied here, these singularities mark the boundary between the region that is influenced by the initial (in this case, leading-edge) conditions and the region influenced by the downstream ($\xi \to \infty$) conditions.

This similarity between the singularities reported here for certain three-dimensional flows and the Stewartson singularity found in certain unsteady two-dimensional flows suggests that, as in the case of the Stewartson singularity, the singularity found in these three-dimensional flows might be removed by employing numerical techniques that account for the direction of the flow of information. In the case of flat plate or wedge set impulsively into motion, the Stewartson singularity is avoided by employing finite differences that have different directional properties depending upon whether α_4 (equation (4)) is positive or negative (Williams & Rhyne 1980). It seems logical that, owing to the above-noted similarity, the singularities that are associated with the Raetz influence principle might be removed if one were to employ finite differences for $\partial W_1/\partial \xi$ and $\partial W_2/\partial \xi$ (equations (14), (15)) according to the sign of the coefficient α_{14} $(= \alpha_{24})$. Employing such a technique simply accounts for the direction of the flow of information from the prescribed boundaries.

Since the outer-edge velocity components are prescribed in boundary-layer theory, the outer-edge streamline pattern is known. It should be a simple matter, then, to determine the streamline at the edge of the boundary layer that defines the limit of the domain of influence for a given set of initial conditions. This line defines two regions in which the numerical derivatives for $\partial W_1/\partial \xi$ and $\partial W_2/\partial \xi$ should be directionally different to account for the direction of the flow of information. On the other hand, the streamline pattern at the wall is not known *a priori*, but is determined as part of the boundary-layer calculation. Thus one can never be sure beforehand that a three-dimensional laminar-boundary-layer calculation will not reach a point where further calculation will violate the Raetz influence principle at the wall. Thus three-dimensional laminar-boundary-layer calculations must be carried out with great care to ensure that the Raetz influence principle is not violated. The present results suggest that if, in a given calculation, the Raetz influence principle is violated, either at the outer edge of the boundary layer or at the wall, one may expect to encounter singular behaviour in the numerical calculation.

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